Three-Dimensional Trajectory Optimization for Aircraft

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A solution method for determining three-dimensional minimum-time aircraft trajectories for aircraft is described. The method is based on Euler-Lagrange optimization theory and energy state approximations. The optimal controls are found by either maximizing or minimizing a modified Hamiltonian. The starting extremum operation and the switching conditions from one extremum operation to the other are determined. The solution is computed by iteration on the two constant adjoint variables. The solution convergence, however, is sensitive to these parameters. A method based on the characteristics of the heading adjoint variable is used to reduce this sensitivity. A roll angle chattering solution, which occurs during the minimum power energy loss segment of the trajectory and causes computational problems, is removed by including a penalty on the roll angle magnitude in the performance criterion. A number of specific problems are then solved.

Nomenclature

a = normal acceleration
c_{d0} = parasite drag coefficient
c_l = lift coefficient
$c_{l\alpha}$ = lift curve slope
D = drag
E = specific energy
g = gravity
h = altitude
k = induced drag coefficient
L = lift
Q = dynamic pressure
S = wing area
T = thrust
t = time
u = control vector
V = velocity
w = weight
x = downrange position
$\boldsymbol{x}$ = state vector
y = cross-range position
z = roll control variable ($z = \tan\phi$)
α = angle of attack
ϕ = roll angle
γ = flight-path angle
λ_E = energy adjoint
λ_x = x position adjoint
λ_ψ = heading adjoint
λ_y = y position adjoint
π = throttle
σ = specific fuel consumption
ψ = heading angle

Introduction

TWO-DIMENSIONAL performance optimization problems for aircraft were addressed and essentially solved in the late 1960s.[1-7]

A three-dimensional optimization problem was first treated by Bryson and Hedrick[15] in 1952 using the energy state approximation. They solved the minimum-time turn to a specified heading. The three-dimensional minimum time to a fixed point problem has only recently received attention in the literature.[8-16] Most approaches use the Euler-Lagrange method and the energy state approximations. Two methods of solution have been developed: one determines the energy adjoint variable from the energy adjoint equation, and the other eliminates the energy adjoint variable. The advantage of the first method is that the controls are found by minimizing a single function. The disadvantage is that three adjoint variables must be determined. The advantage of eliminating the adjoint variable is that the number of adjoint variables that have to be determined are reduced from three to two. The disadvantage is that two possible optimization operations result.

In 1985, Rajan and Ardema[8] addressed the three-dimensional intercept problem using singular-perturbation methods to simplify the problem and the first approach for computing the energy adjoint variable. Thus, they need to solve a two-point boundary-value problem requiring the selection of three adjoint variables. The solution is sensitive to the selection of these variables.

In 1985, Visser et al.[9] addressed the three-dimensional intercept problem using the first method and reduced-order models and also full state models. The reduced-order model equations were solved using backward integration with an assumed value of final heading and values of the energy adjoint. Sensitivity to the choice of the heading adjoint parameter was noted. The full state model was solved with a multiple shooting algorithm. The full state solution requires extensive computations.

In 1981 and 1982, Calise[11,16] used singular-perturbation methods and the second method to address the three-dimensional minimum-time intercept problem. In Ref. 11, he presents two functions for determining the optimal altitude and thrust controls. The functions to be optimized, however, do not contain a necessary restriction on the sign of the denominator. Also, a third function that involves energy loss at maximum thrust is not identified in the solution equations.

A number of papers have addressed the elimination of the energy adjoint for the two-dimensional intercept problems. Schultz and Zagalsky[3] eliminate it using partial derivatives, but the approach is not completely rigorous for on-the-boundary controls. A rigorous solution using partial derivatives is presented by Stein.[4] Erzberger and Lee[5] developed solution equations for the two-dimensional intercept problem without introducing the energy adjoint equation by using energy as the independent variable and separating the solution into climb, cruise, and descent segments. The solution is parameterized in terms of the cruise energy. The two-dimensional results are general and apply to the three-dimensional problem. However, a number of issues remain unresolved. For the planar problem, the switching conditions between

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operations are obvious. This is not the case for the three-dimensional problem. The planar problem requires the determination of only one adjoint variable, and the solution is not sensitive to this parameter, whereas the three-dimensional problem requires a determination of two adjoint variables, and the solution is sensitive to the parameters.

For the three-dimensional problem, these questions remain: Which optimization operation is to be performed first? What are the switching conditions from one operation to the other? How can the sensitivity to the selection of the adjoint variables be reduced? The objective of the paper is to resolve these questions. The approach is: 1) Start with the general full state equations; 2) apply the Euler Lagrange; 3) simplify the equations using the energy state approximations; 4) eliminate the energy adjoint variables using the first integral; 5) derive the switching condition from characteristics of the energy adjoint; and 6) use the characteristics of the heading adjoint variable to develop a low sensitivity iteration method.

Equations of Motion

The equations of motion are

$$\dot{E}=(T\cos\alpha-D)\frac{V}{m},\qquad E=\frac{V^2}{2}+gh \tag{1a}$$

$$\dot{\gamma}=\frac{(L+T\sin\alpha)\cos\phi}{mV}-\frac{g}{V\cos\gamma} \tag{1b}$$

$$\dot{\psi}=\frac{(L+T\sin\alpha)\sin\phi}{mV\cos\gamma} \tag{1c}$$

$$\dot{h}=V\sin\gamma \tag{1d}$$

$$\dot{x}=V\cos\psi\cos\gamma \tag{1e}$$

$$\dot{y}=V\sin\psi\cos\gamma \tag{1f}$$

where thrust T, drag D, and lift L are functions of the form

$$T=T^*(h,V)\pi,\qquad L=Qsc_{l\alpha}\alpha,\qquad D=QS(c_{d0}+kc_l^2) \tag{2}$$

The control variables are throttle setting π, angle of attack α, and roll angle ϕ. The constraints are as follows:

Maximum and minimum acceleration

$$C_1=a-a_{\max}\le 0$$

$$C_2=a_{\min}-a\le 0 \tag{3a}$$

Stall limits

$$C_3=c_l-c_{l\max}\le 0 \tag{3b}$$

Throttle limits

$$C_4=\pi-\pi_{\max}\le 0$$

$$C_5=\pi_{\min}-\pi\le 0 \tag{3c}$$

Altitude limit

$$C_6=H_0-h\le 0 \tag{3d}$$

Placard limit

$$C_7=V-(V_0+ah)\le 0 \tag{3e}$$

These equations can be put into the following matrix form:

$$\dot{x}=f(x,u,t),\qquad x(t_0)=x_0\qquad C(x,u,t)\le 0 \tag{4}$$

Optimization Problem Statement

The general optimization problem can be stated as: Minimize the performance criteria

$$J=\int_0^{t_f}G(x,u,t)\,\mathrm{d}t,\qquad G=1 \tag{5}$$

subject to the constraints

$$\dot{x}=f(x,u,t),\qquad x(t_0)=x_0,\qquad C(x,u,t)\le 0 \tag{6}$$

The solutions to the optimization problem are the Euler-Lagrange equations.

The controls are found by minimizing the Hamiltonian,

$$\min_u H(x,u,t) \tag{7}$$

The Hamiltonian is given by

$$H=G+\lambda^T f \tag{8}$$

The adjoint variables λ are given by

$$\dot{\lambda}=-\frac{\partial H}{\partial x} \tag{9}$$

For H independent of time and for free terminal time, the Hamiltonian must also satisfy

$$H=0 \tag{10}$$

The boundary condition on λ are given by the transversality conditions

$$\left[\lambda^T \mathrm{d}x\right]_{t_0}^{t_f}=0 \tag{11}$$

Simplification of the Euler-Lagrange Equations

The solution to the stated problem is quite difficult to obtain. The equations can be simplified by using singular perturbations as in Refs. 8 and 11, or they can be simplified by using the classical approach of neglecting higher-frequency dynamics. The vertical dynamics can be neglected by assuming that they are of higher frequency than the horizontal dynamics. This assumption is similar to the assumption that the attitude dynamics are of higher frequency than the vertical dynamics, and that the actuator dynamics are of higher frequency than the attitude dynamics. These assumptions ultimately imply that a vertical motion controller, an attitude controller, and an actuator controller are supplied, with the control frequencies of the successively higher dynamics separated by a factor of approximately 10. The neglected equations are replaced with their equilibrium conditions.

$$L\cos\phi=w \tag{12}$$

Also, assume that the angle of attack and flight-path angle are small,

$$\alpha\approx 0,\qquad \gamma\approx 0 \tag{13}$$

Using these assumptions, Eqs. (1) reduce to the energy state equations

$$\dot{E}=(T-D)\frac{V}{m} \tag{14a}$$

$$\dot{\psi}=\frac{g}{V}\tan\phi \tag{14b}$$

$$\dot{x} = V\cos\psi \tag{14c}$$

$$\dot{y} = V\sin\psi \tag{14d}$$

As is well known, the energy state assumptions imply that motion along a constant energy line occurs in zero time and zero distance (zoom climb and dive); i.e., kinetic energy and potential energy can be instantaneously exchanged.

The Hamiltonian reduces to

$$H = 1 + \lambda_E(T-D)\frac{V}{m} + \lambda_\psi\frac{g}{V}\tan\phi + \lambda_x V\cos\psi + \lambda_y V\sin\psi \tag{15}$$

Define

$$z = \tan\phi \tag{16}$$

Using this definition and $L\cos\phi = w$, the drag can be written as

$$D = D_0 + D_c z^2, \qquad D_0 = QSc_{d0} + \frac{kw^2}{QS}, \qquad D_c = \frac{kw^2}{QS} \tag{17}$$

Then, the Hamiltonian becomes

$$H = 1 + \lambda_E(T^*\pi - D_0 - D_c z^2)\frac{V}{m} + \lambda_\psi\frac{g}{V}z + \lambda_x V\cos\psi + \lambda_y V\sin\psi \tag{18}$$

The first and second constraints reduce to

$$|z| \le \sqrt{\left(\frac{a_{\max}}{g}\right)^2 - 1}; \qquad |z| \le \sqrt{\left(\frac{qSc_{l\max}}{mg}\right)^2 - 1} \tag{19}$$

The adjoint variable equations are

$$\dot{\lambda}_E = -\frac{\partial H}{\partial E} \tag{20a}$$

$$\dot{\lambda}_\psi = \lambda_x V\sin\psi - \lambda_y V\cos\psi \tag{20b}$$

$$\dot{\lambda}_x = 0 \tag{20c}$$

$$\dot{\lambda}_y = 0 \tag{20d}$$

Following Rajan and Ardema[8] and Calise,[11] the last three equations can be integrated. The solution is

$$\lambda_x = c_x, \qquad \lambda_y = c_y, \qquad \lambda_\psi = c_x y - c_y x + c_\psi \tag{21}$$

Apply the transversatility conditions

$$\lambda_\psi(t_f) = 0 \tag{22}$$

Thus, one constant of integration can be eliminated

$$c_\psi = -c_x y_f + c_y x_f \tag{23}$$

Note that the approximations are only necessary to solve for the control variables from the Hamiltonian and the adjoint differential equations. After the controls are found, the trajectory can be generated from the full state equations using a high-bandwidth vertical motion controller, e.g.,

$$\alpha = \alpha^* + k_h(h - h^*) + k_{\dot{h}}\dot{h} \tag{24}$$

where a^* and h^* are the optimal variables from the energy state solution. Thus, the controls are approximate, but the trajectory is exact for these controls.

An approximate trajectory can be generated with fewer computations from the approximate energy state equations. The resulting trajectory, however, can have discontinuities.

Elimination of the Energy Adjoint

Equations in the preceding form are used by Rajan and Ardema[8] to solve a minimum-time intercept problem. In the preceding form, the solution is obtained from a single minimizing operation to determine the controls and a selection of three adjoint variables c_x, c_y, and $\lambda_E(t_0)$. The advantage of this approach is that only one optimization operation is used; the disadvantage is that three adjoint constants must be determined. Selection of the parameters is difficult because of the number of them and the sensitivity of the solution to these parameters. Because of this sensitivity, even automatic adjoint adjustment methods such as the Newton-Raphson method may not converge. In the following development, the energy adjoint λ_E is eliminated and the sensitivity problem addressed.

The Hamiltonian contains the time-varying adjoint variable λ_E, which must be computed from its adjoint differential equation. However, λ_E can be eliminated from the equations to be solved. Schultz and Zagalsky[3] and Calise[11] eliminate λ_E using partial derivatives, but the approach is not completely rigorous for on-the-boundary controls. A rigorous solution using partial derivatives is presented in Stein.[4] Erzberger and Lee[5] developed solution equations without introducing λ_E and its associated adjoint equation. This is done by using energy as the independent variable and separating the solution into climb, cruise, and descent segments. The solution is parameterized in terms of the cruise energy. The approach from Ref. 5, which does not introduce λ_E, is not used in the following derivations because other conditions, such as the switching condition, are derived from characteristics of λ_E.

Following is a formal elimination of λ_E from the Hamiltonian without using derivatives. By using $H=0$, λ_E can be eliminated from the equations. First, define

$$f_E = (T-D)\frac{V}{m}, \qquad f_x = V\cos\psi$$

$$f_\psi = \frac{g}{V}z, \qquad f_y = V\sin\psi \tag{25}$$

Then, define

$$H_r = G + \lambda_\psi f_\psi + \lambda_x f_x + \lambda_y f_y \tag{26}$$

Then, the Hamiltonian can be expressed as

$$H = H_r + \lambda_E f_E \tag{27}$$

To eliminate λ_E, consider a set of optimizing controls $\boldsymbol{u}^*$ and a set of nonoptimizing controls $\boldsymbol{u}^* + \Delta\boldsymbol{u}$. Then, because $\boldsymbol{u}^*$ minimizes H,

$$H(\boldsymbol{u}^*) \le H(\boldsymbol{u}^* + \Delta\boldsymbol{u}) \tag{28}$$

Substituting for H using Eqs. (26) and (27)

$$\lambda_E f_E(\boldsymbol{u}^*) + H_r(\boldsymbol{u}^*) - \lambda_E f_E(\boldsymbol{u}^* + \Delta\boldsymbol{u}) - H_r(\boldsymbol{u}^* + \Delta\boldsymbol{u}) \le 0 \tag{29}$$

Using $H=0$, λ_E is

$$\lambda_E = -\frac{H_r(\boldsymbol{u}^*)}{f_E(\boldsymbol{u}^*)} \tag{30}$$

Substituting into Eq. (27),

$$\frac{H_r(\boldsymbol{u}^*)}{f_E(\boldsymbol{u}^*)} f_E(\boldsymbol{u}^* + \Delta\boldsymbol{u}) - H_r(\boldsymbol{u}^* + \Delta\boldsymbol{u}) \le 0 \tag{31}$$

Rearranging Eq. (29), this inequality must be satisfied:

$$f_E(\boldsymbol{u}^* + \Delta\boldsymbol{u})\left[\frac{H_r(\boldsymbol{u}^*)}{f_E(\boldsymbol{u}^*)} - \frac{H_r(\boldsymbol{u}^* + \Delta\boldsymbol{u})}{f_E(\boldsymbol{u}^* + \Delta\boldsymbol{u})}\right] \le 0 \tag{32}$$

Define a new function H_a,

$$H_a = \frac{H_r}{f_E} \tag{33}$$

Then, the controls are determined from

$$\min_u \left[H_a(u)\right]; \quad \text{if } f_E > 0 \tag{34a}$$

$$\max_u \left[H_a(u)\right]; \quad \text{if } f_E < 0 \tag{34b}$$

Substituting for H_r and f_E

$$\min_{\alpha,\pi,z} \left[\frac{1 + \lambda_x V \cos\psi + \lambda_y V \sin\psi + \lambda_\psi \frac{g}{V} z}{(T - D_0 - D_c z^2)} \right]_E, \quad \dot{E} > 0 \quad L = w\sqrt{(1+z^2)} \tag{35a}$$

$$\max_{\alpha,\pi,z} \left[\frac{1 + \lambda_x V \cos\psi + \lambda_y V \sin\psi + \lambda_\psi \frac{g}{V} z}{(T - D_0 - D_c z^2)} \right]_E, \quad \dot{E} < 0 \quad L = w\sqrt{(1+z^2)} \tag{35b}$$

The preceding procedure has eliminated the adjoint variable λ_E but has resulted in two operations for finding the optimal controls.

Switching Conditions

The condition for switching from one operation to the other can be determined as follows. The adjoint variable λ_E corresponding to each operation is given by

$$\lambda_{E1} = -\min_{\alpha,\pi,z} (H_a)_E, \quad \dot{E} > 0 \tag{36a}$$

$$\lambda_{E2} = -\max_{\alpha,\pi,z} (H_a)_E, \quad \dot{E} < 0 \tag{36b}$$

Also, λ_E must satisfy the first differential equation in Eq. (20); thus, λ_E must be continuous. Therefore, the switch from one solution to the other occurs when the two adjoints are equal, i.e.,

$$\lambda_{E1} = \lambda_{E2} \tag{37}$$

or when

$$\max(H_a)_{\dot{E}<0} = \min(H_a)_{\dot{E}>0} \tag{38}$$

General Solution Summary

The general solution procedure is shown in Fig. 1. The steps in generating the solution are as follows: 1) Select the two parameters c_x and c_y, and the type of starting operation (maximum or minimum); 2) compute the optimal controls from either the maximum or the minimum of H_a; 3) compute both the maximum and the minimum of H_a, and switch operations when the functions are equal; 4) integrate the state equations; 5) stop the integration at a specified stopping condition; and 6) update the adjoint constants with the terminal misses and change starting operations if required.

The optimization function can be put into the classical Rutowski form by inverting the function and interchanging the maximum and minimum operations. When this is done, however, restrictions must be made on the domain of the function H_r. Positive and negative values of H_r must be considered separately. The solution in inverted form is described in Appendix B. Calise[11] presents the solution in inverted form but does not include the required restrictions on H_r. Also, the full throttle descent arc is not identified in the paper as a possible solution, and the switching condition between operations is not addressed.

The control determined by optimizing H_a must be consistent with that determined by minimizing the original Hamiltonian. A check for consistency can be made by substituting the values of λ_E determined from optimizing H_a into the original Hamiltonian. On the optimal path, this solution will always be consistent; however, in the process of converging to the optimal path, inconsistent results could occur.

Analytical Solution for Throttle and Roll Angle

In the general solution method, the controls are determined by a search over three control variables α, π, and z. The search over π, however, is quite simple because only two values of π are possible: $\pi = \pi_{max}$ and $\pi = \pi_{min}$. For energy gain flight, only $\pi = \pi_{max}$ is possible. The determination of the controls can be simplified further by analytically solving for z. The simplification will also provide some insight into the nature of the trajectories. First, consider the energy gain solution, then the energy loss solution.

Energy Gain Flight ($\dot{E} > 0$)

The energy gain solution is given by the first part of Eq. (35). Consider the two cases in which the numerator is positive, then negative.

Numerator Positive ($H_r > 0$)

For $\dot{E} > 0$ and $H_r > 0$, the function is minimized by making the denominator a large positive number; thus, the function is minimized with respect to π by $\pi = \pi_{max}$.

An analytical solution for z off the limits can be determined by first taking the partial derivative of the original Hamiltonian with respect to z. This results in

$$z = \frac{\lambda_\psi mg}{2\lambda_E D_c V_2} \tag{39}$$

Substituting into $H = 0$

$$\lambda_E^2 (T - D_0) \frac{V}{m} + \lambda_E (1 + \lambda_x V \cos\psi + \lambda_y V \sin\psi) + \frac{[\lambda_\psi (g/V)]^2}{4\lambda_E D_c (V/m)} = 0 \tag{40}$$

The two roots (λ_{E1}, λ_{E2}) for this expression can be found using the binomial formula, where λ_{E1} corresponds to the negative sign in front of the radical and λ_{E2} to the positive sign. Furthermore, these roots correspond to the λ_E values in Eq. (36). The corresponding values of z are

$$z_1 = \frac{\lambda_\psi mg}{2\lambda_{E1} D_c V^2}; \quad z_2 = \frac{\lambda_\psi mg}{2\lambda_{E2} D_c V^2} \tag{41}$$

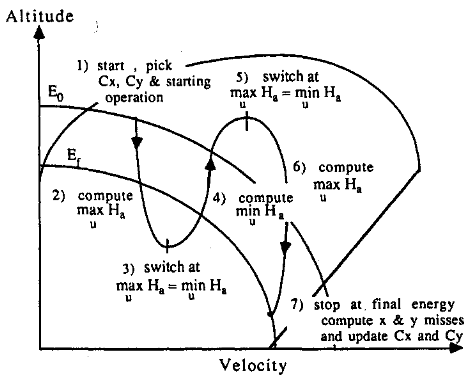


Fig. 1 General solution characteristics.

One of the z solutions corresponds to the energy gain solution. The other corresponds to the energy loss solution. The corresponding z solution can be identified by trying each value. The one that optimizes the function is identified with that solution type. This procedure results in the z_1 solution being identified with the energy gain solution and the z_2 with the energy loss solution. Note that these solutions fail for $\lambda_E = 0$; thus, it may not always be possible to eliminate the z search.

The function with the search over π and z eliminated is

$$\min_{\alpha}\left[\frac{H_r}{(T\pi_{\max}-D)(V/m)}\right], \qquad H_r>0, \quad \dot{E}>0 \qquad z=z_1, \quad L=w\sqrt{(1+z^2)} \tag{42}$$

Numerator Negative ($H_r<0$)

This case, if it exists, provides the smallest value of H_a because H_a will be negative. The function is minimized with respect to throttle by an intermediate value of throttle such that $T-D=0$. However, this is an inconsistent conclusion because $\lambda_E>0$ and the original Hamiltonian is minimized by $\pi=\pi_{\min}$. But for $T=T_{\min}$, $\dot{E}<0$. Thus, on the optimal path, this case cannot exist. If this condition ($H_r<0$, $\dot{E}>0$) occurs during searching for the optimal solution, the adjoint variables have been selected improperly (not close enough to the optimal values), or the wrong starting solution has been selected.

Energy Loss Flight ($\dot{E}<0$)

The energy loss solution is given by the second part of Eq. (35). Consider the numerator negative and positive cases.

Numerator Negative ($H_r<0$)

The value of this function after minimization is negative; thus, this solution is always preferred unless it does not exist. If $H_r<0$ and $\dot{E}<0$, then the function is maximized by making the denominator as small a negative number as possible. This occurs for $\pi=\pi_{\max}$. Intermediate values of π are ruled out by the following consistency argument. The value of λ_E for this case is $\lambda_E<0$. For $\lambda_E<0$, the original Hamiltonian is minimized with respect toπ by $\pi=\pi_{\max}$.

The analytical solution for z not on the boundary is given by the z_2 root. The solution with the π and z search eliminated is

$$\max_{\alpha}\left[\frac{H_r}{(T^*\pi_{\max}-D)(V/m)}\right]_E, \qquad H_r<0, \quad \dot{E}<0 \qquad z=z_2, \; L=w\sqrt{(1+z^2)} \tag{43}$$

This solution occurs in regions where the drag is greater than the maximum thrust. This can occur at large roll angles or at high velocity, low altitude conditions. This arc is a full-powered energy loss arc.

Numerator Positive ($H_r>0$)

If $H_r>0$ and $\dot{E}<0$, then the function is maximized by $\pi=\pi_{\min}$. This arc is a minimum-powered descent.

$$\max_{\alpha,z}\left[\frac{H_r}{(T^*\pi_{\min}-D)(V/m)}\right]_E, \qquad H_r>0, \quad \dot{E}<0 \qquad L=w\sqrt{(1+z^2)} \tag{44}$$

The analytical solution for z for this case needs special examination. Consider the planar unpowered minimum-time descent problem ($\psi=0$, $\psi_f=0$ $\pi_{\min}=0$). The controls are determined from

$$\max_{\alpha,z}\left[\frac{1-(V/V_s)}{(-D_0-D_cz^2)(V/m)}\right]_E, \qquad \dot{E}<0; \quad H_r>0 \qquad L=w\sqrt{(1+z^2)} \tag{45}$$

If $H_r>0$, this function is maximized by choosing $z=z_{\max}$ or $z=z_{\min}$. However, if z continues at one of its extreme values, the solution will not follow a straight line. Thus, $z=z_{\max}$ or $z=z_{\min}$ is not the solution. The only possible solution is a chattering solution in which z chatters infinitely fast between its maximum and minimum values and the trajectory continues in a straight line. This chattering solution, however, will cause computational difficulties on a digital computer because a divergent solution will be generated. Thus, this chattering solution must be eliminated.

Removal of Chattering Solution

The chattering solution can be removed by including a penalty on the roll angle magnitude in the cost function.

$$G=1+k_z z^2 \tag{46}$$

The roll control variable z can still be found from a quadratic equation or its limiting values. This cost function should be used when the chattering solution is likely to occur, that is, when the desired final energy is lower than the initial energy.

Redefinition of Parameters

The parameters c_x and c_y can be expressed in term of parameters that have physical meaning. As discussed in Vinh,[10] the values of c_x and c_y can be expressed as

$$c_x=C_0\cos\psi_f, \qquad c_y=C_0\sin\psi_f \tag{47}$$

For correct dimensionality C_0 must be proportional to $1/V$; therefore, choose C_0 to be

$$C_0=-\frac{1}{V_s}$$

ψ_f is the final heading. For vertical plane motion, V_s is the velocity at which the switch from energy climb to energy descent occurs. λ_ψ is

$$\lambda_\psi=-\frac{1}{V_s}\left[\sin\psi_f(x_f-x)-\cos\psi_f(y_f-y)\right] \tag{48}$$

From Eq. (41), the roll angle is related to λ_ψ. For $\lambda_\psi=0$, $\phi=0$; thus, $\lambda_\psi=0$ defines a roll angle switching line in the x-y plane as shown in Fig. 2. On one side of the line, the roll angle is positive; on the other side, it is negative. The final heading ψ_f must be chosen such that the trajectory ends on the switching line. The solution, however, is sensitive to the selection of ψ_f. In some cases, ψ_f must be determined to three decimal places (e.g., $\psi_f=50.036$ deg). If it is not accurately selected, the solution diverges away from this line.

Reduction of Adjoint Parameter Sensitivity

The approach to reducing this sensitivity is illustrated in Fig. 3. The fundamental idea in reducing this sensitivity is to determine the point of closest approach of λ_ψ to the switching

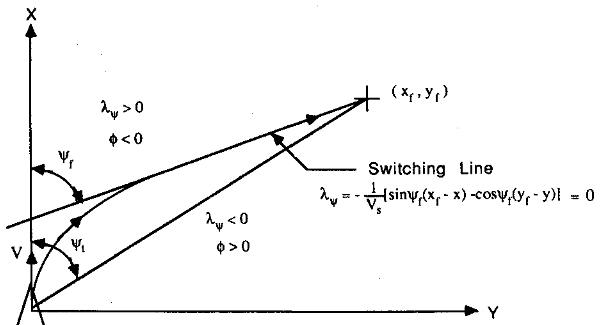


Fig. 2 Roll angle switching line.

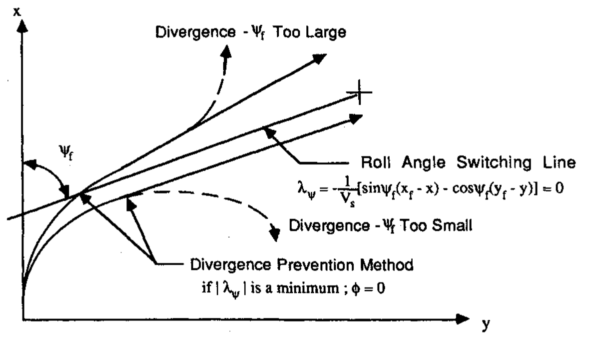


Fig. 3 Divergence and divergence prevention method.

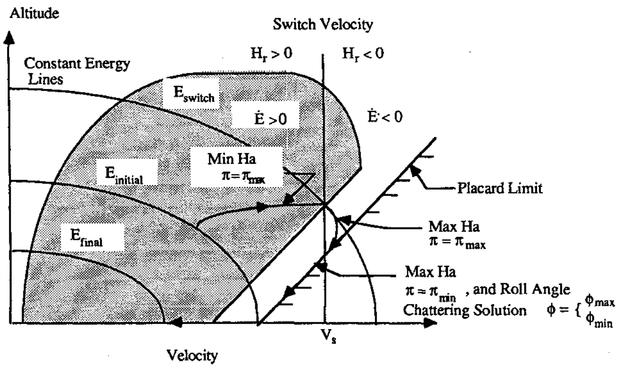


Fig. 4 Minimum time to a fixed position—planar flight.

line ($\lambda_\psi = 0$). When this point is reached, the roll angle is set to zero. The trajectory then continues in a straight line. After the trajectory is computed, the x and y misses can be used to adjust the parameters V_s and ψ_f. A Newton-Raphson method could be employed to update these adjoint constants. The logic used to reduce ψ_f sensitivity is

$$\text{if}|\lambda_\psi(t)| > |\lambda_\psi(t-\Delta t)|, \qquad \phi = 0 \tag{49}$$

Example Solutions

Following is a discussion of the solutions for various cases.

Minimum Time to a Fixed Position—Planar Flight

The objective in treating the planar-minimum time problem is to show the effects of the three-dimensional terms. In this example, the aircraft has the placard limit outside the flight envelope. The final energy is lower than the initial energy. The characteristic of the planar minimum-time intercept is shown in Fig. 4. The adjoint parameter ψ_f is set to 0; V_s is selected to hit the specified terminal range. The first segment of the flight is generated by the minimizing operation. The numerator (H_r) of the function is positive and $\dot{E}>0$; thus, $\pi = \pi_{\max}$. This segment continues until the energy level at which V_s intersects the flight envelope ($H_r = 0$). At this point, the maximum and minimum operations are equal, and the operations are switched. The next flight segment is an energy descent generated by the maximization operation. Here $H_r < 0$ and $\dot{E} < 0$; thus, $\pi = \pi_{\max}$. The trajectory continues along the placard constraint until $V < V_s$; at this point, $H_r > 0$; thus, the solution switches to a minimum throttle. In this region, the function is optimized with respect to z by rapid oscillation between $z = z_{\max}$ and $z = z_{\min}$, i.e., a roll angle chattering solution ($\phi = \pm\phi_{\max}$) while flight continues in a straight line (i.e., $\psi = 0$, $\lambda_\psi = 0$).

The three-dimensional solution is different from the classical planar minimum-time solution, which contains only a full-throttle energy climb and a minimum-throttle energy descent with zero roll angle throughout the flight (see Ref. 3). The physical explanation for the different three-dimensional solution is as follows. Full throttle and, consequently, high energy and high velocities are maintained as long as possible, because high velocities decrease time to the endpoint. After the switch to minimum throttle, drag is increased as much as possible to decrease energy rapidly so that less time is spent at low velocities. High oscillatory roll angles increase the drag because a high angle of attack is required to maintain vertical equilibrium. The chattering solution can be removed as discussed.

Note the existence of a full-power energy loss segment.

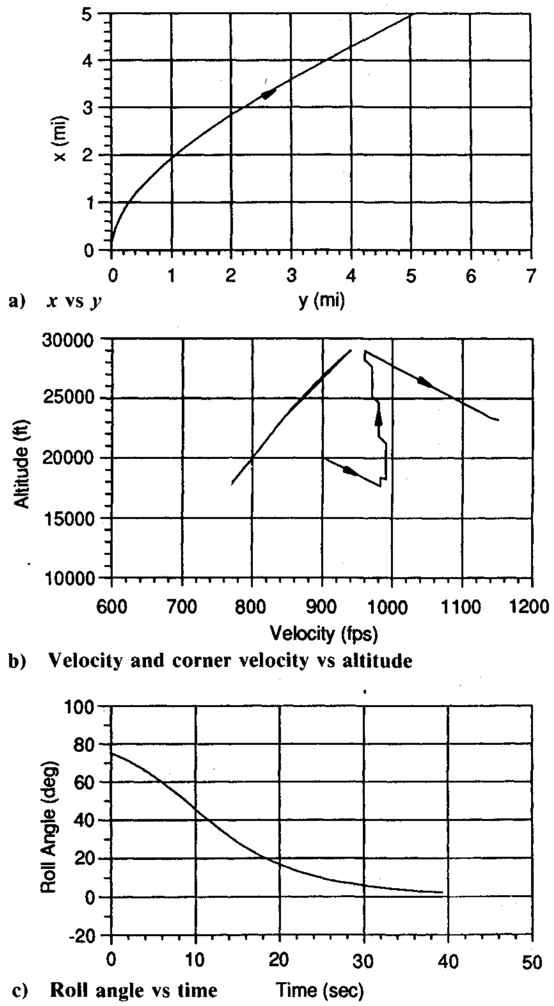
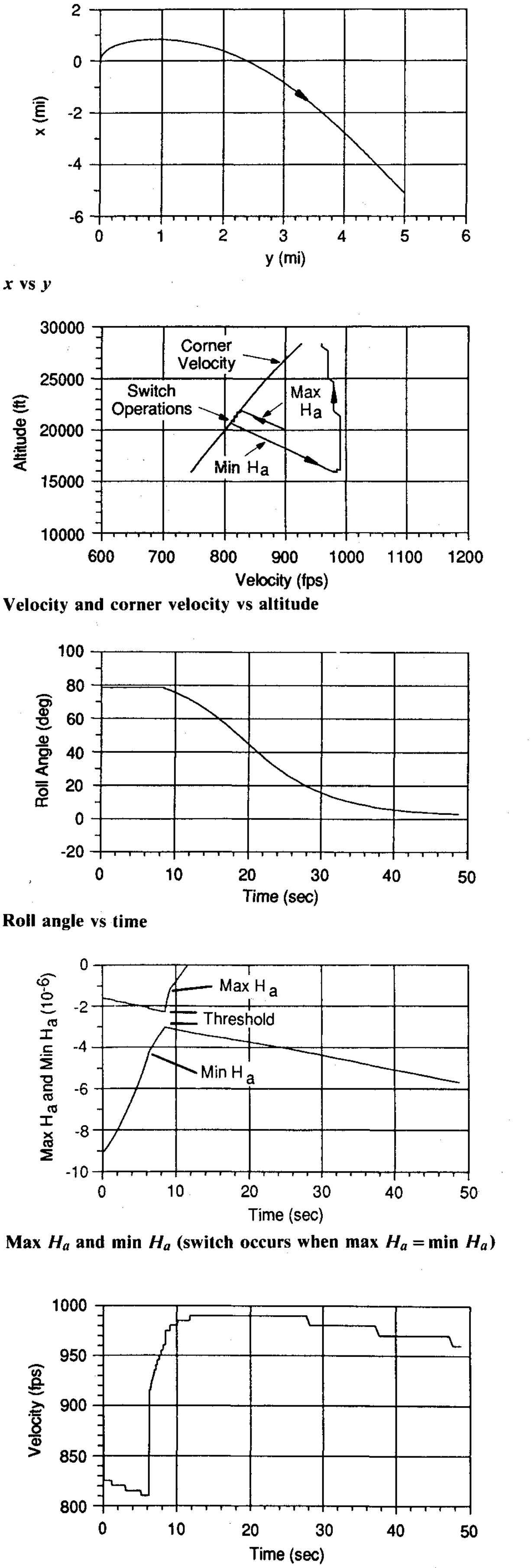


Fig. 5 Minimum time to a fixed position at a small angle (45 deg) off boresight.

Minimum Time to a Fixed Position at a Small Angle (45 deg) off Boresight

For this case, the endpoint is at $x_f = 5$ miles, $y_f = 5$ miles. The initial angle off boresight is 45 deg. The solution is obtained by starting with the minimizing operation. ψ_f and V_s are determined by iteration. Values of V_s and ψ_f are chosen; then, the trajectory equations are integrated and stopped at the final energy. If the terminal positions are not met, new values of ψ_f and V_s are chosen. The process is continued until the terminal position is met. Values of ψ_f greater than the initial angle off are required. Correct values are related to the angle off, range, and aircraft turn radius. Results for this case are shown in Fig. 5. The energy climb occurs at velocities greater than the corner velocity near the maximum energy climb velocity. The throttle is at maximum and the energy rate is positive. The roll angle is less than the maximum acceleration and stall limit and decreases proportionally to zero at the endpoint. The maxi-



a) x vs y

b) Velocity and corner velocity vs altitude

c) Roll angle vs time

d) Max H_a and min H_a (switch occurs when max H_a = min H_a)

e) Velocity vs time

Fig. 6 Minimum time to a fixed position at a high angle (145 deg) off boresight.

mizing operation is not required in the solution. The irregularities in the lines plotted are due to the quantization in the search for the controls.

Minimum Time to a Fixed Position at a Large Angle (145 deg) off Boresight

The terminal position for this case is $x_f = -5$ miles, $y_f = 5$ miles. The initial angle off the velocity vector is 145 deg. The solution is illustrated in Fig. 6. The solution cannot be obtained by using the minimization operation solution alone. Both solution types must be used. The solution is found by first maximizing the function. Maximizing the function at negative energy rate generates flight at the corner velocity. The corner velocity is the velocity at which the stall acceleration intersects the maximum acceleration limit and is also the velocity at which the turn rate is maximum and the turn radius minimum. The throttle is at maximum and the energy rate is negative during flight at the corner velocity. The roll angle is at its limit either on the maximum acceleration limit or the stall limit.

Only part of the turn occurs at the corner velocity. A switch occurs when the maximum and minimum operations are equal (Fig. 6c). After 6.3 s, control is switched to the minimizing operations. Minimizing the function at positive energy rates generates flight at velocities near the maximum energy climb velocity. The roll angle reduces from its maximum value to zero at the endpoint.

Thus, for this case, the plane initially zoom-climbs to the corner velocity and performs a high-rate turn while losing energy. Then the aircraft zoom-dives to a higher velocity and rolls out of the turn and eventually ends in a straight-line course directed at the endpoint. Note that the zoom dive is in inverted flight. This optimal solution has the characteristics of a high-speed yo-yo.

Conclusions

A method of computing optimal three-dimensional aircraft trajectories is determined. The Euler-Lagrange method is used to solve the problem. The Euler-Lagrange equations are simplified using the energy state approximations. After simplification, the controls are found by optimizing a modified Hamiltonian containing two constant adjoint variables. The Hamiltonian is maximized in one region and minimized in another. The switching conditions from one region to the other are determined.

The solution is computed by iteration on the two constant adjoint variables. The convergence of the solution, however, is sensitive to the values of the parameters. An iteration method based on the characteristics of the heading adjoint variable is used to reduce this sensitivity.

A computationally divergent roll angle chattering solution, which occurs during the minimum-power descent portion of the trajectory, is removed by including a penalty on roll angle magnitude in the performance criterion.

A simple analytical expression for determining the roll angle is derived, Also, as a result, determination of the altitude is reduced to a one-dimensional search.

This solution approach has been used to solve a number of cases. For a three-dimensional minimum time to a fixed position at a small angle off the boresight, only the minimizing operation is used. The velocities are near the two-dimensional maximum energy climb velocity. No flight occurs at the corner velocity. The roll angle is lower than the stall limit and the maximum acceleration limit and decreases proportionally to zero at the endpoint. Iteration on two constant parameters is required: one parameter controls the final energy and the other the final heading.

For a three-dimensional minimum time to a fixed position at a high angle off the velocity vector, the function is first maximized and later minimized. Maximizing the function results in a velocity at the corner velocity, maximum turn rate, and negative energy rate. Then the function is minimized,

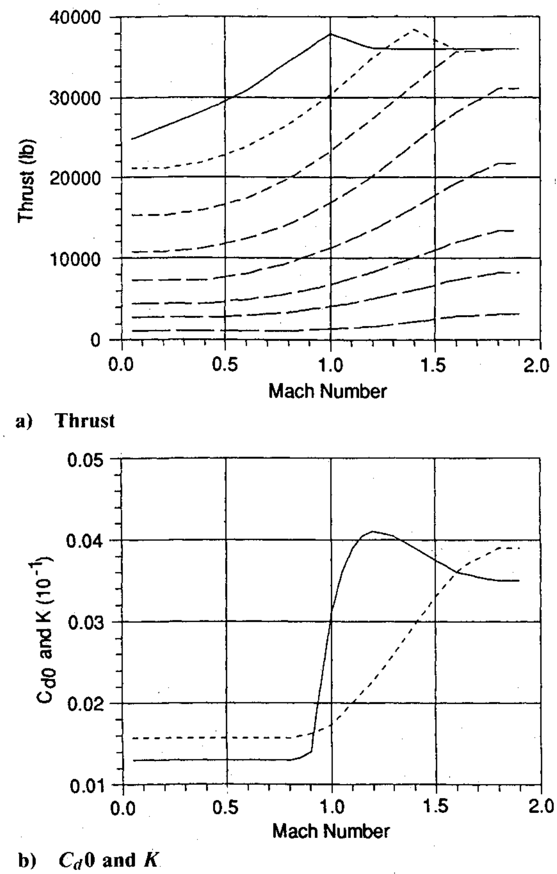


Fig. A1 Aircraft thrust and drag characteristics.

which results in flight at a higher velocity near the two-dimensional maximum energy climb velocity, low turn rates, and positive energy rate. The roll angle begins at the maximum acceleration roll angle limit, when the velocity is at the corner velocity, and then proportionally reduces to zero at the endpoint. This optimal solution is similar to a high-speed yo-yo.

Appendix A: Aircraft Characteristics

The aircraft engine and aerodynamic characteristics are taken from Ref. 1. The thrust and drag coefficients are shown in Fig. A1. The other vehicle parameters are $w = 42{,}000$ lb, $s = 530$ ft^2, $\sigma = 1/1600$ s, $C_{l\max} = 0.945$, and $a_{\max} = 5g$.

Appendix B: The Modified Hamiltonian Function in Inverted Form

The optimization function can be put into the classical Rutowski form by inverting the function and interchanging the maximum and minimum operations. When this is done, however, restrictions must be made on the domain of the function H_r. Positive and negative values of H_r must be considered separately. The solution in inverted form is as follows.

Energy Gain Flight ($\dot{E} > 0$)

In inverted form, the operation for determining the optimal controls during energy gain is

$$\max_u \left[\frac{(T-D)(V/m)}{H_r} \right]; \qquad H_r < 0 \tag{B1}$$

The denominator must be held less than zero during this operation. If the preceding solution does not exist because of constraints, the controls are found from

$$\max_u \left[\frac{(T-D)(V/m)}{H_r} \right]; \qquad H_r > 0 \tag{B2}$$

The denominator must be held positive during this operation.

Energy Loss Flight ($\dot{E} < 0$)

In inverted form, the operation for determining the optimal control during energy loss is

$$\min_u \left[\frac{(T-D)(V/m)}{H_r} \right]; \qquad H_r < 0,\ \dot{E} < 0 \tag{B3}$$

or, if the preceding solution doesn't exist, by

$$\min_u \left[\frac{(T-D)(V/m)}{H_r} \right]; \qquad H_r > 0;\ \dot{E} < 0 \tag{B4}$$

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